Problem 8.5.2

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02407 Stochastic Processes

We consider the position process of a particle $\{S_t\}_{t\geq 0}$ and its associated velocity process $\{V_t\}_{t\geq 0}$. We model the velocity process with an Ornstein-Uhlenbeck process and assume that $S_0 = V_0 = 0$, i.e. the particle is initiated at point zero in a resting position. We then seek the covariance between the position and the velocity processes. By definition, we can write

$$\text{Cov}(V_t, S_t) = \mathbb{E}[V_t S_t] - \mathbb{E}[V_t]\mathbb{E}[S_t].$$

According to eq. (8.58), we have that $\mathbb{E}[V_t | V_0 = 0] = 0$, which simplifies the above to

$$\text{Cov}(V_t, S_t) = \mathbb{E}[V_t S_t].$$

We continue by applying eq. (8.64) with $S_0 = 0$, which leads to

$$\mathbb{E}[V_t S_t] = \mathbb{E}\left[V_t \left(\int_0^t V_u du\right)\right].$$

Recall that the expectation operator is actually an integral operator. Therefore, under certain conditions, we can use Fubini's theorem to interchange the order of integration, which gives us

$$\mathbb{E}\left[V_t \left(\int_0^t V_u du\right)\right] = \int_0^t \mathbb{E}[V_t V_u] du.$$

As stated earlier, $\mathbb{E}[V_t | V_0 = 0] = 0$, and consequently,

$$\mathbb{E}[V_t V_u] = \text{Cov}(V_t, V_u) + \mathbb{E}[V_t]\mathbb{E}[V_u] = \text{Cov}(V_t, V_u).$$

The covariance function of the Ornstein-Uhlenbeck process is given in eq. (8.61), and hence

$$\begin{aligned}\text{Cov}(V_t, S_t) = \mathbb{E}[V_t S_t] &= \int_0^t \mathbb{E}[V_t V_u] du = \int_0^t \text{Cov}(V_t, V_u) du \\ &= \int_0^t \frac{\sigma^2}{2\beta}\left(e^{-\beta(t-u)} - e^{-\beta(t+u)}\right) du.\end{aligned}$$

In order to evaluate the integral, we use a little trick and both multiply and divide by β:

$$\begin{aligned}\text{Cov}(V_t, V_u) &= \frac{\sigma^2}{2\beta^2}\int_0^t \left(\beta e^{-\beta(t-u)} - \beta e^{-\beta(t+u)}\right) du \\ &= \frac{\sigma^2}{2\beta^2} e^{-\beta t}\int_0^t \left(\beta e^{\beta u} - \beta e^{-\beta u}\right) du \\ &= \frac{\sigma^2}{2\beta^2} e^{-\beta t}\left(e^{\beta t} + e^{-\beta t} - 2\right) \\ &= \left(\frac{\sigma}{\beta}\right)^2 e^{-\beta t}\left(\cosh(\beta t) - 1\right).\end{aligned}$$